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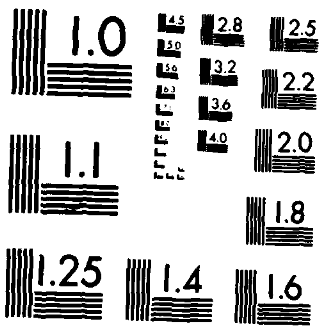
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Variability of Measures of Weapons Effectiveness

Volume IV: Application to Blast Sensitive Targets in the Presence of Delivery Error

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An expression for the probability of kill, P_{kb}, of a stationary ground point-target is derived subject to the following assumptions: the target is killed solely because of blast; the aiming errors of the air-delivered weapon in both range and deflection are independently and normally distributed; the extent of damage to the target is given by a piecewise linear function of the distance between the target and the point of impact. An expression for the variance of P_{kb} is presented. Numerical examples and four appendices are provided.

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Application to Blast Sensitive Targets in the Presence of Delivery Error

PREFACE

This report describes work done by Dr B. D. Sivazlian and Dr J. F. Mahoney, Department of Industrial and Systems Engineering, the University of Florida, Gainesville, Florida 32611 under Contract No. F08635-83-C-0202 sponsored by the Air Force Armament Laboratory (AFATL). The program manager was Mr Daniel A. McInnis (DLYW).

The work was initiated under a 1982 USAF-SCEEE Summer Faculty Research Program sponsored by the Air Force Office of Scientific Research conducted by the Southeastern Center for Electrical Engineering Education (SCEEE) under Contract No. F49620-82-C-0035.

This work addresses itself to the problem of computing the uncertainty associated with the probability of kill P_{kb} due to blast in the presence of aiming error and in the absence of fragmentation. Let P_k be the probability of kill due to blast alone. The assumption is made that (1) P_k is unity between the center of the blast and a distance A from the center, (2) P_k is negligible beyond a certain distance B from the center $(B > A)$, (3) P_k decreases linearly between A and B, (4) the aiming errors in the direction of range and deflection are unbiased, independent of each other having a Gaussian distribution with respective standard deviations σ_x and σ_y. It is shown that P_{kb} can be expressed mathematically by integrals involving the modified Bessel function of the first kind $I_0(\cdot)$. Moreover, under the assumption that the input parameters A, B, σ_x and σ_y are not known with certainty, but are characterized by their first two moments, explicit approximate expressions are obtained for $E[P_{kb}]$ and $Var[P_{kb}]$.

The author has benefited from helpful discussions with several people. Particular thanks are due to Mr Jerry P. Bass, Mr Daniel A. McInnis and Mr Charles A. Reynolds, all from DLYW, who have read the report and have contributed to it through helpful comments.

The report is the fourth of a series dealing with the uncertainty associated with various weapon effectiveness indices and details methodologies and techniques used in computing such uncertainties in the presence of error in the input parameters.

The Public Affairs Office has reviewed this report, and it is releasable to the National Technical Information Service (NTIS), where it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Milton D. Kingcaid

MILTON D. KINGCAID, Colonel, USAF
Chief, Analysis and Strategic Defense Division

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SECTION I

INTRODUCTION

This report considers the problem of estimating the probability of kill due to blast, P_{kb}, in the presence of aiming error. Prior to solving the estimation problem, it is necessary to compute a mathematical expression for P_{kb} in order to apply the usual statistical techniques to arrive at confidence intervals for P_{kb}. In Section II, the derivation of P_{kb} is carried out based on several explicitly stated assumptions. In Section III, the estimation problem for the case when range aiming error equals deflection aiming error is solved. In Section IV, the general estimation problem is accomplished when these two aiming errors are unequal. Finally, Section V provides some conclusive remarks.

No particular problem is encountered in Section III when the two aiming errors are equal. It will be seen that P_{kb} is expressible in terms of the error function. The estimates $E[P_{kb}]$ and $Var[P_{kb}]$ can be obtained in closed form, and again will involve the error function. However, it will be seen that the most general expression for P_{kb}, when $\sigma_x \neq \sigma_y$, involves integrals of the modified Bessel function of zero order $I_0(\cdot)$. Although this result is interesting in itself, nevertheless, it becomes difficult to overcome the mathematical intricacies introduced by the presence of this function. To arrive at closed form expressions for the mean and variance of P_{kb}, in the general case, it will be necessary to use approximation techniques which result in rather lengthy expressions for the estimators.

SECTION II

MATHEMATICAL MODEL FOR P_{kb}

1. Background

In this section, a mathematical model for the probability of kill due to blast, P_{kb}, in the presence of aiming error is developed. The basic situation that one is facing consists of the following.

A weapon whose main effect is kill due to blast is delivered from air to a target point located on the ground surface. The target's position is stationary. The weapon may not hit directly the target due to the presence of aiming errors. These errors are assumed to be unbiased; that is, centered at the location of the target. The target may or may not be killed by the effect of blast depending on whether the weapon impacts close to or away from the target. The probability of kill due to blast is related to the distance between weapon and target by a well-defined mathematical function. In addition, the aiming error is not known precisely but is expressed by a probability density function which provides a mathematical formula for computing the probability that the weapon will impact in an interval du dv close to a point (u,v) on the ground surface.

The technique that will be used to compute the expression for the probability of kill due to blast, P_{kb}, in the presence of aiming error is based on the laws of conditional probability. Ultimately, P_{kb} is not going to depend on the position of the target if the aiming error is unbiased, and if weapon delivery can theoretically result in a point of impact which can be anywhere on the ground surface. On the other hand, P_{kb} will depend on:

a. The parameters specifying the functional form relating probability of kill to distance.

b. The statistical parameters of the aiming error distribution.

2. Assumptions

The following assumptions pertaining to this situation will be made:

a. Both the target and the weapon are idealized as points, and the weapon is aimed at the target.

b. The direction of the weapon delivery range and deflection are, respectively, parallel to the x and y coordinate axes on the ground plane. Since the coordinate system can be arbitrarily selected, there is no loss in generality in making this specific assumption. The position of the target has coordinates (x,y).

c. The aiming error (distance) in each of the x and y directions are independently and normally distributed with respective means x and y and standard deviation σ_x and σ_y. Let (du dv) be the infinitesimal rectangle, close to the point (u,v) at which the weapon impacts, and define the random variables U and V which measure, respectively, the distances between the target point and the weapon impact point along the abscissa and the ordinate. Then, the probability that the weapon will impact in the rectangle (du dv) is:

$$f_{U,V}\ (u-x,v-y)\ du\ dv = \frac{1}{2\pi\ \sigma_x \sigma_y} \exp\left[-\frac{(u-x)^2}{2\sigma_x^2}\right] \times$$

$$\exp\left[-\frac{(v-y)^2}{2\sigma_y^2}\right] du\ dv\ . \qquad (1)$$

d. The probability of kill due to blast at a point (x,y) given that the weapon impacts at (u,v) depends only on the distance r between (x,y) and (u,v) or

$$r = \sqrt{(x-u)^2 + (y-v)^2}$$

and is given by

$$P_k(r) = \begin{cases} 1 & 0 \le r < A \\ \frac{B-r}{B-A} & A < r < B \\ 0 & B < r < \infty \end{cases} \quad (2)$$

Thus, if the target is located at a distance less than A from the center of the blast, the probability of kill is unity. If the target is located at a distance exceeding B, (B > A), the damage to the target is negligible and the probability of kill is zero. Between A and B, the probability of kill is a linear function of the distance. If the delivery of the weapon results in a direct hit on the target (r=0), the probability of kill is unity.

e. Fragmentation effect is neglected. This implies that the target is not fragment sensitive. Or, if weapon fragmentation exists, its effect on the target is negligible, hence, not resulting in a kill.

f. In general, the point in space from which weapon is delivered is nonstationary, the weapon is subject to ballistic errors, etc. It shall be assumed that all these factors combine into a single source of error which is incorporated in the aiming error.

g. The blast does not contribute to the aiming error.

3. Expression for P_{kb}

The probability of kill due to blast is

$$P_{kb} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\text{Probability of kill at } (x,y) \mid \text{weapon impacts at } (u,v)] \times$$

$$[\text{Probability that the weapon impacts between } (u,v) \text{ and } (u+du,\ v+dv)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_k\left[\sqrt{(x-u)^2 + (y-v)^2}\right] \cdot f_{U,V}(u-x,\ v-y)\ du\ dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_k[\sqrt{(x-u)^2 + (y-v)^2}] \frac{1}{2\pi\, \sigma_x \sigma_y} \exp[-\frac{(u-x)^2}{2\sigma_x^2}] \times$$

$$\exp[-\frac{(v-y)^2}{2\sigma_y^2}] \, du\, dv \,. \qquad (3)$$

In the (x,y) plane, translate the axis in such a way that the origin coincides with (x,y), which is the location of the target. Then letting

$$X = u-x \qquad \text{and} \qquad Y = v-y$$

one obtains

$$P_{kb} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_k\,(\sqrt{X^2 + Y^2})\, \frac{1}{2\pi\, \sigma_x \sigma_y} \exp[-(\frac{X^2}{2\sigma_x^2} + \frac{Y^2}{2\sigma_y^2})] \, dX\, dY \,. \qquad (4)$$

Changing now to polar coordinates, let $X = r\cos\theta$ and $Y = r\sin\theta$. This yields

$$P_{kb} = \int_0^{2\pi} \int_0^{\infty} P_k(r)\, \frac{1}{2\pi\, \sigma_x \sigma_y} \exp[-(\frac{r^2\cos^2\theta}{2\sigma_x^2} + \frac{r^2\sin^2\theta}{2\sigma_y^2})] \, r\, dr\, d\theta \,. \qquad (5)$$

Substituting for the value of $P_k(r)$ as given in (2), one obtains

$$P_{kb} = \frac{1}{2\pi\, \sigma_x \sigma_y} \int_0^{2\pi} \int_0^{A} \exp[-(\frac{r^2\cos^2\theta}{2\sigma_x^2} + \frac{r^2\sin^2\theta}{2\sigma_y^2})] \, r\, dr\, d\theta$$

$$+ \frac{1}{2\pi\, \sigma_x \sigma_y} \int_0^{2\pi} \int_A^{B} (\frac{B-r}{B-A}) \exp[-(\frac{r^2\cos^2\theta}{2\sigma_x^2} + \frac{r^2\sin^2\theta}{2\sigma_y^2})] \, r\, dr\, d\theta \,. \qquad (6)$$

Notice here that one is dealing with integrals of the form

$$K(r) = \int_0^{2\pi} \exp[-(\frac{r^2\cos^2\theta}{2\sigma_x^2} + \frac{r^2\sin^2\theta}{2\sigma_y^2})] \, d\theta \,. \qquad (7)$$

In Appendix A it is shown that

$$K(r) = 2\pi\, e^{-ar^2}\, I_0(br^2) \tag{8}$$

where
$$a = \frac{1}{4}\left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right) \tag{9}$$

$$b = \frac{1}{4}\left(\frac{1}{\sigma_y^2} - \frac{1}{\sigma_x^2}\right) \tag{10}$$

and $I_0(\cdot)$ is a modified Bessel function of zero order. Expression (6) for P_{kb} becomes

$$P_{kb} = \frac{1}{\sigma_x \sigma_y} \int_0^A r\, e^{-ar^2}\, I_0(br^2)\, dr$$

$$+ \frac{1}{\sigma_x \sigma_y} \int_A^B \left(\frac{B-r}{B-A}\right) r\, e^{-ar^2}\, I_0(br^2)\, dr\ . \tag{11}$$

SECTION III

ESTIMATION OF $E[P_{kb}]$ AND $Var[P_{kb}]$ WHEN $\sigma_x = \sigma_y = \sigma$

1. Background

In this section, the estimation problem for P_{kb} will be investigated for the particular case when the range aiming error σ_x is equal to the deflection aiming error σ_y ($\sigma_x = \sigma_y = \sigma$). This case is of practical significance in guided weapons for which the aiming error in all directions is about the same. Moreover, the problem itself allows one to obtain closed form expressions for the estimates $E[P_{kb}]$ and $Var[P_{kb}]$ involving the error function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{w^2}{2}} \, dw. \tag{12}$$

Now, since $I_0(0) = 1$, it follows from (11) that

$$P_{kb} = \frac{1}{\sigma^2} \int_0^A r\, e^{-\frac{r^2}{2\sigma^2}} dr + \frac{1}{\sigma^2} \int_A^B \left(\frac{B-r}{B-A}\right) r\, e^{-\frac{r^2}{2\sigma^2}} dr .$$

This expression can be reduced into three integrals as follows:

$$P_{kb} = \frac{1}{\sigma^2} \int_0^A e^{-\frac{r^2}{2\sigma^2}} r\, dr + \frac{B}{B-A} \cdot \frac{1}{\sigma^2} \int_A^B e^{-\frac{r^2}{2\sigma^2}} r\, dr$$

$$- \frac{1}{B-A} \cdot \frac{1}{\sigma^2} \int_A^B r^2 e^{-\frac{r^2}{2\sigma^2}} dr$$

$$= \int_0^A - d\left(e^{-\frac{r^2}{2\sigma^2}}\right) + \frac{B}{B-A} \int_A^B -d\left(e^{-\frac{r^2}{2\sigma^2}}\right) + \frac{1}{B-A} \int_A^B r\, d\left(e^{-\frac{r^2}{2\sigma^2}}\right) .$$

Noting that the two first integrands are exact differentials and integrating by parts the last integral, one obtains

$$P_{kb} = 1 - e^{-\frac{A^2}{2\sigma^2}} + \frac{B}{B-A}\left(e^{-\frac{A^2}{2\sigma^2}} - e^{-\frac{B^2}{2\sigma^2}}\right)$$

$$+ \frac{1}{B-A}\left[r\, e^{-\frac{r^2}{2\sigma^2}}\Big|_A^B - \int_A^B e^{-\frac{r^2}{2\sigma^2}} dr\right]$$

$$= 1 - e^{-\frac{A^2}{2\sigma^2}} + \frac{B}{B-A} e^{-\frac{A^2}{2\sigma^2}} - \frac{B}{B-A} e^{-\frac{B^2}{2\sigma^2}}$$

$$+ \frac{B}{B-A} e^{-\frac{B^2}{2\sigma^2}} - \frac{A}{B-A} e^{-\frac{A^2}{2\sigma^2}} - \frac{1}{B-A}\int_A^B e^{-\frac{r^2}{2\sigma^2}} dr$$

which upon simplification results in

$$P_{kb} = 1 - \frac{1}{B-A}\int_A^B e^{-\frac{r^2}{2\sigma^2}} dr. \tag{13}$$

Let $x = r/\sigma$, then

$$P_{kb} = 1 - \frac{\sigma}{B-A}\int_{\frac{A}{\sigma}}^{\frac{B}{\sigma}} e^{-\frac{x^2}{2}} dx$$

which may be written as

$$P_{kb} = 1 - \frac{\sqrt{2\pi}\,\sigma}{(B-A)}\left[\frac{1}{\sqrt{2\pi}}\int_0^{\frac{B}{\sigma}} e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}}\int_0^{\frac{A}{\sigma}} e^{-\frac{x^2}{2}} dx\right].$$

Using (12) one finally obtains

$$P_{kb} = 1 - \frac{\sqrt{2\pi}\,\sigma}{(B-A)}\left[\Phi\left(\frac{B}{\sigma}\right) - \Phi\left(\frac{A}{\sigma}\right)\right]. \tag{14}$$

It will be implicitly assumed that the parameters A, B, and σ are not known exactly, but are subjected to estimation error. Further, A, B, and σ will be

assumed to be mutually independent and defined by their respective means $\bar{A}$, $\bar{B}$, $\bar{\sigma}$ and their respective variances Var[A], Var[B], and Var[σ].

Recalling that P_{kb} in (14) is a function of the three parameters A, B, and σ, one may write (14) as

$$P_{kb}(A,B,\sigma) = 1 - \frac{\sqrt{2\pi}\ \sigma}{(B-A)} \left[\Phi\left(\frac{B}{\sigma}\right) - \Phi\left(\frac{A}{\sigma}\right)\right] . \tag{15}$$

Let $\bar{A}$, $\bar{B}$ and $\bar{\sigma}$ refer, respectively, to the mean of A, B and σ. Expanding P_{kb} about the point $(\bar{A},\bar{B},\bar{\sigma})$ one obtains up to the first order terms

$$P_{kb}(A,B,\sigma) = P_{kb}(\bar{A},\bar{B},\bar{\sigma}) + (A-\bar{A}) \left.\frac{\partial P_{kb}}{\partial A}\right|_{\bar{A},\bar{B},\bar{\sigma}}$$

$$+ (B-\bar{B}) \left.\frac{\partial P_{kb}}{\partial B}\right|_{\bar{A},\bar{B},\bar{\sigma}} + (\sigma-\bar{\sigma}) \left.\frac{\partial P_{kb}}{\partial \sigma}\right|_{\bar{A},\bar{B},\bar{\sigma}} . \tag{16}$$

2. <u>Estimation of $E[P_{kb}]$</u>

Taking expectations on both sides of (16) yields as a first approximation

$$E[P_{kb}(A,B,\sigma)] = P_{kb}(\bar{A},\bar{B},\bar{\sigma}) \tag{17}$$

or using (15)

$$E[P_{kb}] = 1 - \frac{\sqrt{2\pi}\ \bar{\sigma}}{(\bar{B}-\bar{A})} \left[\Phi\left(\frac{\bar{B}}{\bar{\sigma}}\right) - \Phi\left(\frac{\bar{A}}{\bar{\sigma}}\right)\right] . \tag{18}$$

Let

$$U(A,B,\sigma) = \frac{\sqrt{2\pi}}{B-A} \left[\Phi\left(\frac{B}{\sigma}\right) - \Phi\left(\frac{A}{\sigma}\right)\right] . \tag{19}$$

Then

$$E[P_{kb}] = 1 - \bar{\sigma}\ U(\bar{A},\bar{B},\bar{\sigma}) . \tag{20}$$

3. Estimation of $\mathrm{Var}[P_{kb}]$

First, expression (16) is written as

$$P_{kb}(A,B,\sigma) - P_{kb}(\bar{A},\bar{B},\bar{\sigma}) = (A-\bar{A}) \left.\frac{\partial P_{kb}}{\partial A}\right|_{\bar{A},\bar{B},\bar{\sigma}}$$

$$+ (B-\bar{B}) \left.\frac{\partial P_{kb}}{\partial B}\right|_{\bar{A},\bar{B},\bar{\sigma}} + (\sigma-\bar{\sigma}) \left.\frac{\partial P_{kb}}{\partial \sigma}\right|_{\bar{A},\bar{B},\bar{\sigma}} . \qquad (21)$$

It was assumed that each of the random variables A, B, and σ is mutually independent. Thus, squaring and taking the variance on both sides of (21) results in:

$$\mathrm{Var}[P_{kb}] = \left(\frac{\partial P_{kb}}{\partial A}\right)^2_{\bar{A},\bar{B},\bar{\sigma}} \mathrm{Var}[A] + \left(\frac{\partial P_{kb}}{\partial B}\right)^2_{\bar{A},\bar{B},\bar{\sigma}} \mathrm{Var}[B]$$

$$+ \left(\frac{\partial P_{kb}}{\partial \sigma}\right)^2_{\bar{A},\bar{B},\bar{\sigma}} \mathrm{Var}[\sigma] . \qquad (22)$$

To obtain the partial derivatives, form (13) of P_{kb} will be used rather than form (14). First compute $\partial P_{kb}/\partial A$:

$$\frac{\partial P_{kb}}{\partial A} = -\frac{1}{(B-A)^2} \int_A^B e^{-\frac{r^2}{2\sigma^2}} dr + \frac{1}{B-A} e^{-\frac{A^2}{2\sigma^2}}$$

$$= -\frac{\sigma\sqrt{2\pi}}{(B-A)^2} \left[\frac{1}{\sqrt{2\pi}} \int_0^{\frac{B}{\sigma}} e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \int_0^{\frac{A}{\sigma}} e^{-\frac{x^2}{2}} dx\right] + \frac{1}{B-A} e^{-\frac{A^2}{2\sigma^2}}$$

$$= -\frac{\sigma\sqrt{2\pi}}{(B-A)^2} \left[\Phi\left(\frac{B}{\sigma}\right) - \Phi\left(\frac{A}{\sigma}\right)\right] + \frac{1}{B-A} e^{-\frac{A^2}{2\sigma^2}} .$$

Using the definition (19) yields finally

$$\frac{\partial P_{kb}}{\partial A} = -\frac{\sigma}{B-A} U(A,B,\sigma) + \frac{1}{B-A} e^{-\frac{A^2}{2\sigma^2}} . \qquad (23)$$

Next $\partial P_{kb}/\partial B$ is computed:

$$\frac{\partial P_{kb}}{\partial B} = \frac{1}{(B-A)^2}\int_A^B e^{-\frac{r^2}{2\sigma^2}}\,dr - \frac{1}{B-A}\,e^{-\frac{B^2}{2\sigma^2}}$$

$$= \frac{\sigma\sqrt{2\pi}}{(B-A)^2}\left[\frac{1}{\sqrt{2\pi}}\int_0^{\frac{B}{\sigma}} e^{-\frac{x^2}{2}}\,dx - \frac{1}{\sqrt{2\pi}}\int_0^{\frac{A}{\sigma}} e^{-\frac{x^2}{2}}\,dx\right] - \frac{1}{B-A}\,e^{-\frac{B^2}{2\sigma^2}} .$$

Using (19), this expression may be written as

$$\frac{\partial P_{kb}}{\partial B} = \frac{\sigma\sqrt{2\pi}}{(B-A)^2}\left[\Phi\left(\frac{B}{\sigma}\right) - \Phi\left(\frac{A}{\sigma}\right)\right] - \frac{1}{B-A}\,e^{-\frac{B^2}{2\sigma^2}} . \tag{24}$$

Finally, $\partial P_{kb}/\partial \sigma$ is computed. Using (13) yields

$$\frac{\partial P_{kb}}{\partial \sigma} = -\frac{1}{B-A}\int_A^B \frac{r^2}{2}\,(-2)\,\sigma^{-3}\,e^{-\frac{r^2}{2\sigma^2}}\,dr$$

$$= -\frac{1}{(B-A)\,\sigma}\int_A^B -r\left(\frac{r}{\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}}\,dr$$

$$\frac{\partial P_{kb}}{\partial \sigma} = -\frac{1}{(B-A)\,\sigma}\int_A^B r\,d\left(e^{-\frac{r^2}{2\sigma^2}}\right) .$$

Integrating by part results in (see Section III-1):

$$\frac{\partial P_{kb}}{\partial \sigma} = -\frac{1}{(B-A)\,\sigma}\left[B\,e^{-\frac{B^2}{2\sigma^2}} - A\,e^{-\frac{A^2}{2\sigma^2}} - \int_A^B e^{-\frac{r^2}{2\sigma^2}}\,dr\right]$$

$$= \frac{1}{(B-A)\,\sigma}\left[A\,e^{-\frac{A^2}{2\sigma^2}} - B\,e^{-\frac{B^2}{2\sigma^2}}\right]$$

$$+ \frac{\sqrt{2\pi}}{(B-A)}\left[\frac{1}{\sqrt{2\pi}}\int_0^{\frac{B}{\sigma}} e^{-\frac{x^2}{2}}\,dx - \frac{1}{\sqrt{2\pi}}\int_0^{\frac{A}{\sigma}} e^{-\frac{x^2}{2}}\,dx\right].$$

Using the definition (19) gives

$$\frac{\partial P_{kb}}{\partial\sigma} = \frac{1}{(B-A)\,\sigma}\left[A\,e^{-\frac{A^2}{2\sigma^2}} - B\,e^{-\frac{B^2}{2\sigma^2}}\right] + \frac{\sqrt{2\pi}}{B-A}\left[\Phi\left(\frac{B}{\sigma}\right) - \Phi\left(\frac{A}{\sigma}\right)\right] . \qquad (25)$$

Substituting (23), (24), and (25) in (22) yields

$$\mathrm{Var}[P_{kb}] = \left[-\frac{\bar\sigma}{\bar B-\bar A}\,U(\bar A,\bar B,\bar\sigma) + \frac{1}{\bar B-\bar A}\,e^{-\frac{\bar A^2}{2\bar\sigma^2}}\right]^2 \mathrm{Var}[A]$$

$$+ \left[\frac{\bar\sigma}{\bar B-\bar A}\,U(\bar A,\bar B,\bar\sigma) - \frac{1}{\bar B-\bar A}\,e^{-\frac{\bar B^2}{2\bar\sigma^2}}\right]^2 \mathrm{Var}[B]$$

$$+ \left[\frac{1}{(\bar B-\bar A)\,\bar\sigma}\left(\bar A\,e^{-\frac{\bar A^2}{2\bar\sigma^2}} - \bar B\,e^{-\frac{\bar B^2}{2\bar\sigma^2}}\right) + U(\bar A,\bar B,\bar\sigma)\right]^2 \mathrm{Var}[\sigma]. \qquad (26)$$

4. Example

A guided weapon whose main effect is blast is aimed at the center 0 of a target. Because of error, the weapon will not necessarily impact at 0. This error, measured as the distance between the center 0 of the target and the center of impact, is unbiased with mean located at 0 and standard deviation of σ such that $E[\sigma] = 15$ ft and $\mathrm{Var}[\sigma] = \frac{25}{3}$ ft^2. If the weapon impacts at a point which is at a distance less than A ft from 0, the target is killed with probability one. On the other hand, if the weapon impacts at a point distant B ft from 0 $(A < B)$, the target is undamaged. The parameters A and B are estimates and are equally likely to be in the respective intervals $13 < A < 17$ ft and $20 < B < 22$ ft. It is required to provide a two-standard deviation confidence interval for the probability of kill due to blast P_{kb}.

The objective is to calculate $E[P_{kb}]$ and $\mathrm{Var}[P_{kb}]$. It is easy to verify that

$$\bar A = E[A] = 15 \text{ ft} \quad ; \quad \mathrm{Var}[A] = \frac{(17-13)^2}{12} = \frac{4}{3}\text{ ft}^2$$

$$\bar B = E[B] = 21 \text{ ft} \quad ; \quad \mathrm{Var}[B] = \frac{(22-20)^2}{12} = \frac{1}{3}\text{ ft}^2$$

$$\bar\sigma = E[\sigma] = 15 \text{ ft} \quad ; \quad \mathrm{Var}[\sigma] = \frac{25}{3}\text{ ft}^2.$$

First compute $U(\bar{A},\bar{B},\bar{\sigma})$ using (19)

$$U(\bar{A},\bar{B},\bar{\sigma}) = \frac{\sqrt{2\pi}}{(21-15)} [\Phi(\frac{21}{15}) - \Phi(\frac{15}{15})]$$

$$= .417,771\ (.419,243 - .341,345)$$

$$= .032,543.$$

Using (20) it follows that

$$E[P_{kb}] = 1 - \bar{\sigma}\, U(\bar{A},\bar{B},\bar{\sigma})$$

$$= 1 - (15)\ (.032,543)$$

$$= .5119.$$

To compute $Var[P_{kb}]$, each of the variance components in (26) is computed separately. First the component incorporating $Var[A]$;

$$[- \frac{\bar{\sigma}}{(\bar{B}-\bar{A})} U(\bar{A},\bar{B},\bar{\sigma}) + \frac{1}{(\bar{B}-\bar{A})} e^{-\frac{1}{2}\left(\frac{\bar{A}}{\bar{\sigma}}\right)^2}]^2\ Var[A]$$

$$= [- \frac{15}{(21-15)} (.032,543) + \frac{1}{(21-15)} e^{-\frac{1}{2}\left(\frac{15}{15}\right)^2}]^2\ (\frac{4}{3})$$

$$= (- .081,358 + .101,088)^2\ (\frac{4}{3})$$

$$= .000,519.$$

Next, the variance component involving $Var[B]$ is computed.

$$\left[\frac{\bar{\sigma}}{(\bar{B}-\bar{A})} U(\bar{A},\bar{B},\bar{\sigma}) - \frac{1}{(\bar{B}-\bar{A})} e^{-\frac{1}{2}\left(\frac{\bar{B}}{\bar{\sigma}}\right)^2}\right]^2 \mathrm{Var}[B]$$

$$= \left[\frac{15}{(21-15)} (.032{,}543) - \frac{1}{(21-15)} e^{-\frac{1}{2}\left(\frac{21}{15}\right)^2}\right]^2 \left(\frac{1}{3}\right)$$

$$= (.081{,}358 - .062{,}552)^2 \left(\frac{1}{3}\right)$$

$$= .000{,}118 .$$

Finally, the variance component involving Var[σ] is computed. Thus,

$$\left[\frac{1}{(\bar{B}-\bar{A})\,\bar{\sigma}} \left(\bar{A}\, e^{-\frac{1}{2}\left(\frac{\bar{A}}{\bar{\sigma}}\right)^2} - \bar{B}\, e^{-\frac{1}{2}\left(\frac{\bar{B}}{\bar{\sigma}}\right)^2}\right) + U(\bar{A},\bar{B},\bar{\sigma})\right]^2 \mathrm{Var}[\sigma]$$

$$\left[\frac{1}{(21-15)(15)} \left[15\, e^{-\frac{1}{2}\left(\frac{15}{15}\right)^2} - 21\, e^{-\frac{1}{2}\left(\frac{21}{15}\right)^2}\right] + .032{,}543\right]^2 \left(\frac{25}{3}\right)$$

$$= \left[\frac{1}{90} (9.097{,}960 - 7.881{,}533) + .032{,}543\right]^2 \left(\frac{25}{3}\right)$$

$$= .017{,}678 .$$

Hence,

$$\mathrm{Var}[P_{kh}] = .000{,}519 + .000{,}118 + .017{,}678$$

$$= .018{,}315$$

$$\sigma_{P_{kh}} = \sqrt{\mathrm{Var}[P_{kh}]} = .1353$$

$$P_{kh} = E[P_{kh}] \pm 2\,\sigma_{P_{kh}}$$

$$= .5119 \pm .2706 .$$

SECTION IV

ESTIMATION OF E[P_{kb}] AND VAR[P_{kb}] WHEN $\sigma_x \neq \sigma_y$

1. Background

For general purpose bombs, range delivery error σ_x is usually greater than deflection delivery error σ_y. From (11), the expression for the probability of kill due to blast P_{kb} is given by

$$P_{kb} = \frac{1}{\sigma_x \sigma_y} \int_0^A re^{-ar^2} I_0(br^2)\, dr + \frac{1}{\sigma_x \sigma_y} \int_A^B \left(\frac{B-r}{B-A}\right) re^{-ar^2} I_0(br^2)\, dr \qquad (11)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of the zeroth order.

The development of closed form expressions for E[P_{kb}] and Var[P_{kb}] using the result (11) is hindered by the appearance of the integral term involving the function $I_0(\cdot)$. However, it is possible to obtain an approximate expression for P_{kb} by using the expanded form of $I_0(x)$ and retaining the first three terms. The resulting expression is more readily amenable to numerical computation than the original expression (11). Now

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \cdots$$

Substituting this last expression in (11) yields

$$P_{kb} = \frac{1}{\sigma_x \sigma_y} \int_0^A re^{-ar^2} \left(1 + \frac{b^2 r^4}{4} + \frac{b^4 r^8}{64} + \cdots\right) dr + \frac{1}{\sigma_x \sigma_y} \left(\frac{B}{B-A}\right) \int_A^B re^{-ar^2} \left(1 + \frac{b^2 r^4}{4} + \frac{b^4 r^8}{64} + \cdots\right) dr$$

$$- \frac{1}{\sigma_x \sigma_y} \left(\frac{1}{B-A}\right) \int_A^B r^2 \, e^{-ar^2} \left(1 + \frac{b^2 r^4}{4} + \frac{b^4 r^8}{64} + \cdots\right) dr \; . \qquad (27)$$

The result of the evaluation of the integrals in this expression is given in Appendix B. Thus, let

$$J_1(A) = \int_0^A r e^{-ar^2} \, dr$$

$$= \frac{1}{2a} \left(1 - e^{-aA^2}\right) \qquad (28)$$

$$J_2(A) = \int_0^A r^5 \, e^{-ar^2} \, dr$$

$$= \frac{1}{a^3} \left[1 - e^{-aA^2} \left(1 + aA^2 + \frac{a^2 A^4}{2}\right)\right] \qquad (29)$$

$$J_3(A) = \int_0^A r^9 \, e^{-ar^2} \, dr$$

$$= \frac{12}{a^5} \left[1 - e^{-aA^2} \left(1 + aA^2 + \frac{a^2 A^4}{2} + \frac{a^3 A^6}{6} + \frac{a^4 A^8}{24}\right)\right] \qquad (30)$$

$$J_4(A) = \int_0^A r^2 \, e^{-ar^2} \, dr$$

$$= \frac{\sqrt{\pi}}{2a^{3/2}} \, \Phi(A \sqrt{2a}) - \frac{A}{2a} \, e^{-aA^2} \qquad (31)$$

$$J_5(A) = \int_0^A r^6 \, e^{-ar^2} \, dr$$

$$= \frac{15 \sqrt{\pi}}{8a^{7/2}} \, \Phi(A \sqrt{2a}) - \frac{15}{8} \frac{A}{a^3} \, e^{-aA^2} \left(1 + \frac{2}{3} aA^2 + \frac{4}{15} a^2 A^4\right) \qquad (32)$$

$$J_6(A) = \int_0^A r^{10} \, e^{-ar^2} \, dr$$

$$= \frac{945\sqrt{\pi}}{32\, a^{11/12}} \Phi\,(A\sqrt{2a}) - \frac{945}{32}\frac{A}{a^5} e^{-aA^2}\,(1 + \frac{2}{3} aA^2$$

$$+ \frac{4}{15} a^2A^4 + \frac{8}{105} a^3A^6 + \frac{16}{945} a^4A^8) \ . \qquad (33)$$

Expression (27) can then be written as:

$$P_{kb} \simeq \frac{1}{\sigma_x\sigma_y} [J_1(A) + \frac{b^2}{4} J_2(A) + \frac{b^4}{64} J_3(A)]$$

$$+ \frac{1}{\sigma_x\sigma_y} (\frac{B}{B-A}) \{[J_1(B) - J_1(A)] + \frac{b^2}{4} [J_2(B) - J_2(A)]$$

$$+ \frac{b^4}{64} [J_3(B) - J_3(A)]\} - \frac{1}{\sigma_x\sigma_y} (\frac{1}{B-A}) \{[J_4(B) - J_4(A)]$$

$$+ \frac{b^2}{4} [J_5(B) - J_5(A)] + \frac{b^4}{64} [J_6(B) - J_6(A)]\} \ . \qquad (34)$$

It will be assumed that the parameters A, B, σ_x, and σ_y are subject to estimation error. Further A, B, σ_x, and σ_y will be assumed to be mutually independent and characterized by their respective means $\overline{A}$, $\overline{B}$, $\overline{\sigma}_x$, and $\overline{\sigma}_y$ and their respective variances Var[A], Var[B], Var[σ_x], and Var[σ_y].

Recalling that P_{kb} in (11) is a function of the four parameters A, B, σ_x, and σ_y, one may write (11) as

$$P_{kb}(A,B,\sigma_x,\sigma_y) = \frac{1}{\sigma_x\sigma_y} \int_0^A re^{-ar^2}\, I_0(br^2)\, dr$$

$$+ \frac{1}{\sigma_x\sigma_y} \int_A^B (\frac{B-r}{B-A})\, re^{-ar^2}\, I_0(br^2)\, dr \ . \qquad (35)$$

Let $\bar{A}$, $\bar{B}$, $\bar{\sigma}_x$, and $\bar{\sigma}_y$ refer, respectively, to the mean of A, B, σ_x, and σ_y. Expanding P_{kb} about the point ($\bar{A}$, $\bar{B}$, $\bar{\sigma}_x$, and $\bar{\sigma}_y$) one obtains up to the first order terms

$$
\begin{aligned}
P_{kb}(A,B,\sigma_x,\sigma_y) &= P_{kb}(\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y) + (A-\bar{A}) \left.\frac{\partial P_{kb}}{\partial A}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} \\
&+ (B-\bar{B}) \left.\frac{\partial P_{kb}}{\partial B}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} + (\sigma_x-\bar{\sigma}_x) \left.\frac{\partial P_{kb}}{\partial \sigma_x}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} \\
&+ (\sigma_y-\bar{\sigma}_y) \left.\frac{\partial P_{kb}}{\partial \sigma_y}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y}
\end{aligned}
\tag{36}
$$

2. Estimation of $E[P_{kb}]$

Taking expectations on both sides of (36) yields as a first approximation

$$
E[P_{kb}(A,B,\sigma_x,\sigma_y)] = P_{kb}(\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y) \; . \tag{37}
$$

Let

$$
\bar{a} = \frac{1}{4}\left(\frac{1}{\bar{\sigma}_x^2} + \frac{1}{\bar{\sigma}_y^2}\right) \tag{38}
$$

$$
\bar{b} = \frac{1}{4}\left(\frac{1}{\bar{\sigma}_y^2} - \frac{1}{\bar{\sigma}_x^2}\right) \; . \tag{39}
$$

Using [illegible], one obtains as an approximation to (37)

$$
\begin{aligned}
E[P_{kb}] &= \frac{1}{\bar{\sigma}_x\bar{\sigma}_y}\left[J_1(\bar{A}) + \frac{\bar{b}^2}{4} J_2(\bar{A}) + \frac{\bar{b}^4}{64} J_3(\bar{A})\right] \\
&+ \frac{1}{\bar{\sigma}_x\bar{\sigma}_y}\left(\frac{\bar{B}}{\bar{B}-\bar{A}}\right)\left\{[J_1(\bar{B}) - J_1(\bar{A})] + \frac{\bar{b}^2}{4}[J_2(\bar{B}) - J_2(\bar{A})]\right. \\
&\left.+ \frac{\bar{b}^4}{64}[J_3(\bar{B}) - J_3(\bar{A})]\right\} - \frac{1}{\bar{\sigma}_x\bar{\sigma}_y}\left(\frac{1}{\bar{B}-\bar{A}}\right)\left\{[J_4(\bar{B}) - J_4(\bar{A})]\right. \\
&\left.+ \frac{\bar{b}^2}{4}[J_5(\bar{B}) - J_5(\bar{A})] + \frac{\bar{b}^4}{64}[J_6(\bar{B}) - J_6(\bar{A})]\right\} \; .
\end{aligned}
\tag{40}
$$

3. Estimation of $Var[P_{kb}]$

First expression (36) is written as

$$P_{kb}(A,B,\sigma_x,\sigma_y) - P_{kb}(\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y)$$

$$= (A-\bar{A}) \left.\frac{\partial P_{kb}}{\partial A}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} + (B-\bar{B}) \left.\frac{\partial P_{kb}}{\partial B}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y}$$

$$+ (\sigma_x-\bar{\sigma}_x) \left.\frac{\partial P_{kb}}{\partial \sigma_x}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} + (\sigma_y-\bar{\sigma}_y) \left.\frac{\partial P_{kb}}{\partial \sigma_y}\right|_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} . \tag{41}$$

It was assumed that each of the random variables A, B, σ_x, and σ_y are mutually independent. Thus, squaring and taking the variance on both sides of (41) results in

$$Var[P_{kb}] = \left(\frac{\partial P_{kb}}{\partial A}\right)^2_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} Var[A] + \left(\frac{\partial P_{kb}}{\partial B}\right)^2_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} Var[B]$$

$$+ \left(\frac{\partial P_{kb}}{\partial \sigma_x}\right)^2_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} Var[\sigma_x] + \left(\frac{\partial P_{kb}}{\partial \sigma_y}\right)^2_{\bar{A},\bar{B},\bar{\sigma}_x,\bar{\sigma}_y} Var[\sigma_y] . \tag{42}$$

The details of the calculation of the partial derivative terms appear in Appendix C. The final results are

$$\frac{\partial P_{kb}}{\partial A} = \frac{\sigma_x \sigma_y P_{kb} - \int_0^A re^{-ar^2} I_0(br^2)\, dr}{(B-A)\, \sigma_x \sigma_y} \tag{43}$$

$$\frac{\partial P_{kb}}{\partial B} = \frac{\int_0^B re^{-ar^2} I_0(br^2)\, dr - \sigma_x \sigma_y P_{kb}}{(B-A)\, \sigma_x \sigma_y} \tag{44}$$

$$\frac{\partial P_{kb}}{\partial \sigma_x} = - \frac{\int_A^B r^2 e^{-ar^2} \{I_0(br^2) - I_1(br^2)\}\, dr}{2(B-A)\, \sigma_x^2\, \sigma_y} \tag{45}$$

$$\frac{\partial P_{kb}}{\partial \sigma_y} = - \frac{\int_A^B r^2 e^{-ar^2} \left[I_0(br^2) - I_1(br^2)\right] dr}{2(B-A)\, \sigma_x \sigma_y^2} . \qquad (46)$$

Here $I_1(\cdot)$ is the modified Bessel function of the first kind of the first order. In series form one has

$$I_1(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!(k+1)!2^{2k+1}}$$

We may use the same technique as for $E[P_{kb}]$ to evaluate expression (43), (44), (45), and (46) by using $I_0(x)$ and $I_1(x)$ and retaining the first few terms. However, we will show an alternate method which relies on the Chi-Square function for the numerical evaluation of $Var[P_{kb}]$. Let

$$X_n(B) = \int_0^B r^n e^{-ar^2} I_0(br^2)\, dr \qquad (47)$$

$$Y_n(B) = \int_0^B r^n e^{-ar^2} I_1(br^2)\, dr \qquad (48)$$

Then (43) through (46) may be written as

$$\frac{\partial P_{kb}}{\partial A} = \frac{P_{kb}}{B-A} - \frac{X_1(A)}{\sigma_x \sigma_y (B-A)} \qquad (49)$$

$$\frac{\partial P_{kb}}{\partial B} = \frac{X_1(B)}{\sigma_x \sigma_y (B-A)} - \frac{P_{kb}}{B-A} \qquad (50)$$

$$\frac{\partial P_{kb}}{\partial \sigma_x} = - \frac{1}{2\sigma_x^2 \sigma_y (B-A)} \left[X_2(B) - X_2(A) + Y_2(B) - Y_2(A)\right] \qquad (51)$$

$$\frac{\partial P_{kb}}{\partial \sigma_y} = - \frac{1}{2\sigma_x \sigma_y^2 (B-A)} \left[Y_2(B) - Y_2(A) - X_2(B) + X_2(A)\right] \qquad (52)$$

It is possible to express (47) and (48) in terms of

$$P(\chi^2|\nu) = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})}\int_0^{\chi^2} t^{\frac{\nu}{2}-1}\, e^{-\frac{t}{2}}\, dt \qquad 0 \leq \chi^2 < \infty \tag{53}$$

The function $P(\chi^2|\nu)$ refers to the chi-squared probability function and is tabulated in several places (see e.g., [1]). The function is also available as the PROBCHI function provided by the SAS computer program [3]. In Appendix D, it is shown that

$$X_n(U) = \int_0^U r^n\, e^{-ar^2}\, I_0(br^2)\, dr$$

$$= \frac{\Gamma(\frac{n+1}{2})}{2a^{\frac{n+1}{2}}}\Big[P(2aU^2|n+1) + \frac{(\frac{b}{4a})^2\,(n+1)(n+3)}{(1!)^2}\,P(2aU^2|n+5)$$

$$+ \frac{(\frac{b}{4a})^4\,(n+1)(n+3)(n+5)(n+7)}{(2!)^2}\,P(2aU^2|n+9)$$

$$+ \frac{(\frac{b}{4a})^6\,(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)}{(3!)^2}\,P(2aU^2|n+13) + \cdots\Big] \tag{54}$$

and

$$Y_n(U) = \int_0^U r^n\, e^{-ar^2}\, I_1(br^2)\, dr$$

$$= \frac{(\frac{b}{4a})\,\Gamma(\frac{n+1}{2})}{2a^{\frac{n+1}{2}}}\Big[(n+1)\,P(2aU^2|n+3)$$

$$+ \frac{(\frac{b}{4a})^2\,(n+1)(n+3)(n+5)}{1!\,2!}\,P(2aU^2|n+7)$$

$$+ \frac{(\frac{b}{4a})^4\,(n+1)(n+3)(n+5)(n+7)(n+9)}{2!3!}\,P(2aU^2|n+11) + \cdots\Big] \tag{55}$$

4. Example

A general purpose bomb whose main effect is blast is dropped from air and aimed at the center 0 of a target. The range delivery error is 20 ft with a standard deviation of 3 ft. The deflection delivery error is 17 ft with a standard deviation of 2 ft. If the weapon impacts at a point which is at a distance less than A ft from 0, the target is killed with probability one. On the other hand, if the weapon impacts at a point distance B ft from 0 (A < B), the target is undamaged. The parameters A and B are estimates and are equally likely to be in the respective intervals 13 < A < 17 ft and 20 < B < 22 ft. It is required to provide a two-standard deviation confidence interval for the probability of kill due to blast P_{kb}.

Let σ_x and σ_y be the respective range and deflection delivery errors. One has

$$\bar{A} = E[A] = 15 \text{ ft} \quad ; \quad Var[A] = \frac{(17-13)^2}{12} = \frac{4}{3} \text{ ft}^2$$

$$\bar{B} = E[B] = 21 \text{ ft} \quad ; \quad Var[B] = \frac{(22-20)^2}{12} = \frac{1}{3} \text{ ft}^2$$

$$\bar{\sigma}_x = E[\sigma_x] = 20 \text{ ft} \quad ; \quad Var[\sigma_x] = 3^2 = 9 \text{ ft}^2$$

$$\bar{\sigma}_y = E[\sigma_y] = 17 \text{ ft} \quad ; \quad Var[\sigma_y] = 2^2 = 4 \text{ ft}^2.$$

Calculation of $E[P_{kb}]$

$E[P_{kb}]$ is calculated using (40). For convenience, the bar will be dropped from the symbols describing the input parameters.

From (38) and (39), a and b are computed

$$a = \frac{1}{4}\left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right) = \frac{1}{4}\left[\frac{1}{(20)^2} + \frac{1}{(17)^2}\right]$$

$$= .001,490,051,9$$

$$b = \frac{1}{4}\left(\frac{1}{\sigma_y^2} - \frac{1}{\sigma_x^2}\right) = \frac{1}{4}\left[\frac{1}{(17)^2} - \frac{1}{(20)^2}\right]$$

$$= .000,240,051,9 \ .$$

Next, each of the quantities $J_i(A)$ and $J_i(B)$, $i=1,2,\ldots,6$ is computed using expressions (27) through (32)

<u>Computation of $J_1(A)$</u>

$$J_1(A) = \frac{1}{2a}\left(1 - e^{-aA^2}\right)$$

$$= \frac{1}{(2)(.001,490,051,9)}\{1 - \exp[-(.001,490,051,9)(15)^2]\}$$

$$= 95.583,609,22$$

<u>Computation of $J_2(A)$</u>

$$J_2(A) = \frac{1}{a^3}\{1 - e^{-aA^2}\left(1 + aA^2 + \frac{a^2A^4}{2}\right)]$$

$$= \frac{1}{(.001,490,051,9)^3}\{1 - (.715,150,923)\ [1 + .335,261,677,5$$

$$+ \frac{.112,[illegible]}{2}]\}$$

$$= \frac{1}{(.001,490,051,9)^3}[1 - (.715,150,923)(1.391,461,874)]$$

$$= [illegible].4[illegible]9,54[illegible],658 \times 10^6$$

Computation of $J_3(A)$

$$J_3(A) = \frac{12}{a^5}\left[1 - e^{-aA^2}\left(1 + aA^2 + \frac{a^2A^4}{2} + \frac{a^3A^6}{6} + \frac{a^4A^8}{24}\right)\right]$$

$$= \frac{12}{(.001{,}490{,}051{,}9)^5}\left[1 - (.715{,}150{,}923)(1 + .335{,}261{,}677{,}5\right.$$

$$\left. + \frac{.112{,}400{,}392{,}4}{2} + \frac{.037{,}683{,}544{,}1}{6} + \frac{.012{,}633{,}842{,}2}{24})\right]$$

$$= \frac{12 \times 10^{15}}{(149{,}005{,}19)^5}\left[1 - (.715{,}150{,}923)(1.398{,}263{,}875)\right]$$

$$= 4.365{,}863{,}756 \times 10^{10}.$$

Computation of $J_4(A)$

$$J_4(A) = \frac{\sqrt{\pi}}{2a^{3/2}}\,\Phi(A\sqrt{2a}) - \frac{A}{2a}\,e^{-aA^2}$$

$$= \frac{\sqrt{\pi}}{(2)(.001{,}490{,}051{,}9)^{3/2}}\,\Phi\left[(15)\sqrt{(2)(.001{,}490{,}051{,}9)}\right]$$

$$- \frac{(15)(.715{,}150{,}923)}{(2)(.001{,}490{,}051{,}9)}$$

$$J_4(A) = \frac{(\sqrt{\pi})(.293{,}57)}{(2)(.000{,}057{,}517{,}\bar{8})}\;3{,}599.627{,}585$$

$$= 923.661{,}527{,}2$$

Computation of $J_5(A)$

$$J_5(A) = \frac{15\sqrt{\pi}}{8a^{7/2}}\,\Phi(A\sqrt{2a}) - \frac{15}{8}\,\frac{A}{a^3}\,e^{-aA^2}\left(1 + \frac{2}{3}aA^2 + \frac{4}{15}a^2A^4\right)$$

$$= \frac{(15)\sqrt{\pi}}{(8)(.001,490,051,9)^{7/2}} \Phi(.818,854,904,7)$$

$$- \frac{(15)(15)(.715,150,923)}{(8)(.001,490,051,9)^{3}} [1 + \frac{2}{3} (.335,261,677,5)$$

$$+ \frac{4}{15} (.335,261,677,5)^{2}]$$

$$= 7.639,817,718 \times 10^{9} - (6.079,754,57)(1.253,481,223) \times 10^{9}$$

$$= 1.895,952,4 \times 10^{7}$$

<u>Computation of $J_6(A)$</u>

$$J_6(A) = \frac{945\sqrt{\pi}}{32\ a^{11/2}} \Phi(A\sqrt{2a}) - \frac{945}{32} \frac{A}{a^5} e^{-aA^2} [1 + \frac{2}{3} aA^2$$

$$+ \frac{4}{15} a^2A^4 + \frac{8}{105} a^3A^6 + \frac{16}{945} a^4A^8]$$

$$= \frac{945\sqrt{\pi}}{(32)(.001,490,051,9)^{11/2}} \Phi(.818,854,904,7)$$

$$- \frac{(945)(15)(.715,150,923)}{(32)(.001,490,051,9)^{5}} [1 + \frac{2}{3} (.335,261,677,5)$$

$$+ \frac{4}{15} (.112,400,392,4) + \frac{8}{105} (.037,683,544,1)$$

$$+ \frac{16}{945} (.012,633,848,2)]$$

$$= \frac{(945)(1.772,453,851)}{(32)(.001,490,051,9)^{11/2}} (.293,57)$$

$$- \frac{(945)(15)(.715,150,923)}{(32)(.001,490,051,9)^{5}} (1.256,566,259)$$

$$J_6(A) = 54.195,192,55 \times 10^{15} - 54.193,750,53 \times 10^{15}$$

$$= 1.442,016,6 \times 10^{12} .$$

Computation of $J_1(B)$

$$J_1(B) = \frac{1}{2a}\,(1 - e^{-aB^2})$$

$$= \frac{1}{(2)(.001{,}490{,}051{,}9)}\,\{1 - \exp[-(.001{,}490{,}051{,}9)(21)^2]\}$$

$$= 161.623{,}330{,}6\ .$$

Computation of $J_2(B)$

$$J_2(B) = \frac{1}{a^3}\,[1 - e^{-aB^2}\,(1 + aB^2 + \frac{a^2B^4}{2})]$$

$$= \frac{1}{(.001{,}490{,}051{,}9)^3}\,[1 - (.518{,}345{,}698{,}4)(1 + .657{,}112{,}887{,}9$$

$$+ \frac{.431{,}797{,}3474}{2})]$$

$$= 8.805{,}900{,}503 \times 10^6$$

Computation of $J_3(B)$

$$J_3(B) = \frac{12}{a^5}\,[1 - e^{-aB^2}\,(1 + aB^2 + \frac{a^2B^4}{2} + \frac{a^3B^6}{6} + \frac{a^4B^8}{24})]$$

$$= \frac{12}{(.001{,}490{,}051{,}9)^5}\,[1 - (.518{,}345{,}698{,}4)(1 + .657{,}112{,}887{,}9$$

$$+ \frac{.431{,}797{,}347{,}4}{2} + \frac{.28{,}373{,}960{,}2}{6} + \frac{.186{,}448{,}949{,}3}{24})]$$

$$= \frac{12\times10^{15}}{(1.490{,}051{,}9)^5}\,[1 - (.518{,}345{,}698{,}4)(1.928{,}070{,}201)]$$

$$= 96.896{,}317{,}28 \times 10^{10}\ .$$

Computation of $J_4(B)$

$$J_4(B) = \frac{\sqrt{\pi}}{2a^{3/2}}\,\Phi(B\sqrt{2a}) - \frac{B}{2a}\,e^{-aB^2}$$

$$= \frac{\sqrt{\pi}}{(2)(.001{,}490{,}051{,}9)^{3/2}}\,\Phi[(21)\sqrt{(20)(.001{,}490{,}051{,}9}]$$

$$- \frac{(21)(.518,345,698,4)}{(2)(.001,490,051,9)}$$

$$J_4(B) = \frac{\sqrt{\pi}}{(2)(.000,057,517,8)} \Phi(1.146,396,867) - 3,652.644,47$$

$$= 5,765.317,71 - 3,652.644,47$$

$$= 2,112.673,24 \ .$$

<u>Computation of $J_5(B)$</u>

$$J_5(B) = \frac{15\sqrt{\pi}}{8a^{7/2}} \Phi(B\sqrt{2a}) - \frac{15}{8}\frac{B}{a^3} e^{-aB^2} \left(1 + \frac{2}{3} aB^2 + \frac{4}{15} a^2B^4\right)$$

$$= \frac{15\sqrt{\pi}}{8(.001,490,051,9)^{7/2}} \Phi(1.146,396,867)$$

$$- \frac{15}{8} \frac{(21)(.518,345,698,4)}{(.001,490,051,9)^3} \left[1 + \frac{2}{3}(.657,112,887,9)\right.$$

$$\left. + \frac{4}{15}(.431,797,347,4)\right]$$

$$= 9.737,599,188 \times 10^9 - 9.582,287,423 \times 10^9$$

$$= 15.531,176,54 \times 10^7 \ .$$

<u>Computation of $J_6(B)$</u>

$$J_6(B) = \frac{945\sqrt{\pi}}{32\ a^{11/2}} \Phi(B\sqrt{2a}) - \frac{945}{32}\frac{B}{a^5} e^{-aB^2} \left[1 + \frac{2}{3} aB^2\right.$$

$$\left. + \frac{4}{15} a^2B^4 + \frac{8}{105} a^3B^6 + \frac{16}{945} a^4B^8\right]$$

$$= \frac{945\sqrt{\pi}}{(32)(.001,490,051,9)^{5.5}} \Phi(1.146,396,867)$$

$$- \frac{945}{32} \frac{(21)}{(.001,490,051,9)^5} (.518,345,698,4)$$

$$\left[1 + \frac{2}{3}(.657,112,887,9) + \frac{4}{15}(.431,797,347,4)\right.$$

$$+ \frac{8}{105} (.283,739,602) + \frac{16}{945} (.186,448,949,3)]$$

$$= 69.076,394,55 \times 10^{15} - (43.763,661,05) (10^{15}) (1.577,996,281)$$

$$= (69.076,394,55 - 69.058,894,38) \times 10^{15}$$

$$= 17.500,174,5 \times 10^{12} .$$

To compute $E[P_{kb}]$, the numerical values of σ_x, σ_y, b, $J_i(A)$ and $J_i(B)$ (i=1,...,6) are substituted in (40). To simplify the computation, the three major terms of (40) are computed separately.

$$\text{Term 1} = \frac{1}{\sigma_x \sigma_y} [J_1(A) + \frac{b^2}{4} J_2(A) + \frac{b^4}{64} J_3(A)]$$

$$= \frac{1}{(17)(20)} [95.583,609,22 + \frac{(.000,240,051,9)^2}{4} .$$

$$(1.479,540,658) \times 10^6 + \frac{(.000,240,051,9)^4}{64} (4.365,863,756) \times 10^{10}]$$

$$= \frac{1}{340} [95.583,609,22 + .021,314,601 + .000,002,265,2]$$

$$= .281,19 .$$

$$\text{Term 2} = \frac{1}{\sigma_x \sigma_y} (\frac{B}{B-A}) \{[J_1(B) - J_1(A)] + \frac{b^2}{4} [J_2(B) - J_2(A)]$$

$$+ \frac{b^4}{64} [J_3(B) - J_3(A)]\} .$$

$$= \frac{1}{(17)(20)} (\frac{21}{21-15}) [(161.623,330,6 - 95.583,609,22)$$

$$+ \frac{(.000,240,051,9)^2}{4} (8.805,900,503 - 1.479,540,658) \times 10^6$$

$$+ \frac{(.000,240,051,9)^4}{64} (96.896,017,28 - 4.365,863,756) \times 10^{10}]$$

$$= \frac{(21)}{(17)(20)(6)} [66.039,721,38 + \frac{(2.400,519)^2(7.326,359,845)}{(4)(100)}$$

$$+ \frac{(2.400,519)^4(92.530,153,52)}{(64)(10^6)}]$$

$$= \frac{(21)}{(17)(20)(6)} (66.039,721,38 + .105,545,215,3 + .000,048,003,1)$$

$$= \frac{(21)(66.145,314,59)}{(17)(20)(6)}$$

$$= .68091 \ .$$

$$\text{Term 3} = \frac{1}{\sigma_x \sigma_y} \left(\frac{1}{B-A}\right) \{[J_4(B) - J_4(A)] + \frac{b^2}{4} [J_5(B) - J_5(A)]$$

$$+ \frac{b^4}{64} [J_6(B) - J_6(A)]\}$$

$$= \frac{1}{(17)(20)} \left(\frac{1}{21-15}\right) [(2,112.673,24 - 923.661,527,2)$$

$$+ \frac{(.000,240,051,9)^2}{4} (15.531,176,54 - 1.895,952,4) \times 10^7$$

$$+ \frac{(.000,240,051,9)^4}{64} (17.500,174,5 - 1.442,016,6) \times 10^{12}]$$

$$= \frac{1}{(17)(20)(6)} [1,189.011,713 + \frac{(2.400,519)^2(13.635,224,14)}{(4)(10)}$$

$$+ \frac{(2.400,519)^4(16.058,157,9)}{(64)(10^4)}]$$

$$= \frac{1}{(17)(20)(6)} (1,189,011,713 + 1,964,321,57 + .000,833,175,2)$$

$$= \frac{1,190,976,868}{(17)(20)(6)} = .583,81 \ .$$

$$E[P_{kb}] = \text{Term 1} + \text{Term 2} - \text{Term 3}$$

$$= .281,19 + .680,91 - .583,81$$

$$= .378,29 \ .$$

Calculation of $Var[P_{kb}]$

To compute $Var[P_{kb}]$, the following procedure is used:

a. Retaining only the first two terms of the series (54) and (55) one finds from the tables (e.g., [1]) using interpolation the following

$$P(2aA^2|2) = 0.284,843,021,8$$

$$P(2aA^2|3) = 0.119,900,605,6$$

$$P(2aA^2|5) = 0.015,452,739,7$$

$$P(2aA^2|6) = 0.004,893,986,9$$

$$P(2aA^2|7) = 0.001,442,416,9$$

$$P(2aA^2|9) = 0.000,109,195,8$$

Similarly, one finds

$$P(2aB^2|2) = 0.481,647,535,2$$

$$P(2aB^2|3) = 0.274,248,026,6$$

$$P(2aB^2|5) = 0.066,542,951,0$$

$$P(2aB^2|6) = 0.029,130,319,2$$

$$P(2aB^2|7) = 0.011,944,485,8$$

$$P(2aB^2|9) = 0.000,109,195,8$$

b. Substituting these values in (54) and (55) one obtains for $n=1,2$ the following:

$$X_1(A) = 95.602,888,58$$

$$X_1(B) = 161.747,910,3$$

$$X_2(A) = 923.977,344,6$$

$$X_2(B) = 2,115.029,112$$

$$Y_2(A) = 14.428,854,99$$

$$Y_2(B) = 61.985,977,35$$

c. These values are next substituted in (49), (50), (51), and (52) to yield:

$$\frac{\partial P_{kb}}{\partial A} = 0.016,175,955$$

$$\frac{\partial P_{kb}}{\partial B} = 0.016,248,075,3$$

$$\frac{\partial P_{kb}}{\partial \sigma_x} = -0.015,179,030,5$$

$$\frac{\partial P_{kb}}{\partial \sigma_y} = -0.016,486,370,3 \ .$$

4. Using now (42) results in

$$\begin{aligned} \mathrm{Var}[P_{kb}] &= (0.016,175,955)^2 \left(\tfrac{4}{3}\right) + (0.016,248,075,3)^2 \left(\tfrac{1}{3}\right) \\ &\quad + (-0.015,179,030,5)^2 (9) + (-0.016,486,370,3)^2 (4) \\ &= 0.003,597,710,3 \ . \end{aligned}$$

Hence,

$$\sigma_{P_{kb}} = 0.059{,}98 \text{ .}$$

Thus, $$P_{kb} = E[P_{kb}] \pm 2\,\sigma_{P_{kb}}$$

$$= .378{,}29 \pm .109{,}96$$

SECTION V

CONCLUSION

In this report, the probability of kill due to blast of an exploding weapon is related to the aiming error for the weapon in order to obtain a closed form expression for the net probability of kill due to blast, P_{kb}.

Using the Taylor's series estimation procedure, expressions for $E[P_{kb}]$ and $Var[P_{kb}]$ are obtained in order to arrive at two standard deviation confidence levels on P_{kb}. Since closed form expressions for $E[P_{kb}]$ and $Var[P_{kb}]$ are not possible to derive, several approximation techniques are used to obtain numerical results.

REFERENCES

[1] Abramowitz, M. and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc., N.Y., 1965.

[2] Gradshteyn, I. S. and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, N.Y., 1965.

[3] SAS User's Guide, SAS Institute Inc., P.O. Box 10066, Raleigh, N.C. 27605, 1979.

APPENDIX A

EVALUATION OF K(r)

The intent is to evaluate the integral

$$K(r) = \int_0^{2\pi} \exp[-(\frac{r^2\cos^2\theta}{2\sigma_x^2} + \frac{r^2\sin^2\theta}{2\sigma_y^2})]\ d\theta \ .$$

Using the trigonometric formulas

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

gives for K(r)

$$K(r) = \exp[-\frac{r^2}{4}(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2})] \int_0^{2\pi} \exp[\frac{r^2}{4}(\frac{1}{\sigma_y^2} - \frac{1}{\sigma_x^2}) \cos 2\theta]\ d\theta \ .$$

Let $a = \frac{1}{4}(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2})$

$$b = \frac{1}{4}(\frac{1}{\sigma_y^2} - \frac{1}{\sigma_x^2}) \ .$$

Then

$$K(r) = e^{-ar^2} \int_0^{2\pi} e^{br^2\cos 2\theta}\ d\theta \ .$$

In K(r), let $2\theta = x$, then

$$K(r) = \frac{1}{2} e^{-ar^2} \int_0^{4\pi} e^{br^2 \cos x}\, dx$$

$$= \frac{1}{2} e^{-ar^2} \left[\int_0^{2\pi} + \int_{2\pi}^{4\pi}\right] e^{br^2 \cos x}\, dx \ .$$

In the second integral, let $x = 2\pi + y$, then

$$K(r) = \frac{1}{2} e^{-ar^2} \left[\int_0^{2\pi} e^{br^2 \cos x}\, dx + \int_0^{2\pi} e^{br^2 \cos y}\, dy\right]$$

$$= e^{-ar^2} \int_0^{2\pi} e^{br^2 \cos x}\, dx$$

$$= e^{-ar^2} \left[\int_0^{\pi} + \int_{\pi}^{2\pi}\right] e^{br^2 \cos x}\, dx \ .$$

In the second integral, let $x = \pi + y$, then

$$K(r) = e^{-ar^2} \left[\int_0^{\pi} e^{br^2 \cos x}\, dx + \int_0^{\pi} e^{br^2 \cos(\pi+y)}\, dy\right.$$

$$= e^{-ar^2} \left[\int_0^{\pi} e^{br^2 \cos x}\, dx + \int_0^{\pi} e^{-br^2 \cos y}\, dy\right]$$

But, from [2], it is found that the following holds:

$$\int_0^{\pi} e^{\pm br^2 \cos y}\, dy = \pi\, I_0(br^2)$$

where $I_0(\cdot)$ is the modified Bessel function of zero order. Hence,

$$K(r) = 2\pi\, e^{-ar^2}\, I_0(br^2) \ .$$

APPENDIX B

EVALUATION OF $J_1(A)$ THROUGH $J_6(A)$

In this appendix, expression (26) which is rewritten is to be evaluated.

$$P_{kb} = \frac{1}{\sigma_x\sigma_y} \int_0^A re^{-ar^2} \left(1 + \frac{b^2r^4}{4} + \frac{b^4r^8}{64} + \cdots\right) dr$$

$$+ \frac{1}{\sigma_x\sigma_y} \left(\frac{1}{B-A}\right) \int_A^B re^{-ar^2}\left(1 + \frac{b^2r^4}{4} + \frac{b^4r^8}{64} + \cdots\right) dr$$

$$- \frac{1}{\sigma_x\sigma_y} \left(\frac{1}{B-A}\right) \int_A^B r^2 e^{-ar^2} \left(1 + \frac{b^2r^4}{4} + \frac{b^4r^8}{64} + \cdots\right) dr \ . \qquad (B-1)$$

It is clear that this expression is reducible to the evaluation of six integrals, namely

$$J_1(A) = \int_0^A re^{-ar^2} dr \quad ; \qquad J_2(A) = \int_0^A r^5 e^{-ar^2} dr$$

$$J_3(A) = \int_0^A r^9 e^{-ar^2} dr \quad ; \qquad J_4(A) = \int_0^A r^2 e^{-ar^2} dr$$

$$J_5(A) = \int_0^A r^6 e^{-ar^2} dr \quad ; \qquad J_6(A) = \int_0^A r^{10} e^{-ar^2} dr \ .$$

Each of these integrals will be evaluated separately.

1. Evaluation of $J_1(A) = \int_0^A re^{-ar^2} dr$

$$J_1(A) = \int_0^A re^{-ar^2} dr = -\frac{1}{2a} \int_0^A d(e^{-ar^2})$$

$$= -\frac{1}{2a} e^{-ar^2}\Big|_0^A = \frac{1}{2a} (1 - e^{-aA^2}) \ . \qquad (B-2)$$

2. Evaluation of $J_2(A) = \int_0^A r^5 e^{-ar^2} dr$

$$J_2(A) = \int_0^A r^5 e^{-ar^2} dr = -\frac{1}{2a}\int_0^A r^4 d(e^{-ar^2})$$

$$= -\frac{1}{2a}\left[e^{-ar^2} r^4\Big|_0^A - \int_0^A 4r^3 e^{-ar^2} dr\right]$$

$$= -\frac{A^4}{2a} e^{-aA^2} + \frac{2}{a}\int_0^A r^3 e^{-ar^2} dr$$

$$= -\frac{A^4}{2a} e^{-aA^2} + \frac{2}{a}\left(-\frac{1}{2a}\right)\int_0^A r^2 d(e^{-ar^2})$$

$$= -\frac{A^4}{2a} e^{-aA^2} - \frac{1}{a^2}\left[e^{-ar^2} r^2\Big|_0^A - \int_0^A 2re^{-ar^2} dr\right]$$

$$= -\frac{A^4}{2a} e^{-aA^2} - \frac{A^2}{a^2} e^{-aA^2} + \frac{2}{a^2}\left(-\frac{1}{2a}\right)\int_0^A d(e^{-ar^2})$$

$$= -\frac{A^4}{2a} e^{-aA^2} - \frac{A^2}{a^2} e^{-aA^2} - \frac{1}{a^3}\ e^{-ar^2}\Big|_0^A$$

$$= -\frac{A^4}{2a} e^{-aA^2} - \frac{A^2}{a^2} e^{-aA^2} - \frac{1}{a^3} e^{-aA^2} + \frac{1}{a^3}$$

$$= \frac{1}{a^3}\left[1 - e^{-aA^2}\left(1 + aA^2 + \frac{a^2A^4}{2}\right)\right] . \qquad (B\text{-}3)$$

3. Evaluation of $J_3(A) = \int_0^A r^9 e^{-ar^2} dr$

$$J_3(A) = \int_0^A r^9 e^{-ar^2} dr = -\frac{1}{2a}\int_0^A r^8 d(e^{-ar^2})$$

$$= -\frac{1}{2a}\left[e^{-ar^2} r^8\Big|_0^A - \int_0^A 8r^7 e^{-ar^2} dr\right]$$

$$J_3(A) = -\frac{A^8}{2a} e^{-aA^2} + \frac{4}{a}\int_0^A r^7 e^{-ar^2}\, dr$$

$$= -\frac{A^8}{2a} e^{-aA^2} + \frac{4}{a}\left(-\frac{1}{2a}\right)\int_0^A r^6\, d(e^{-ar^2})$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2}{a^2}\left[e^{-ar^2} r^6\Big|_0^A - \int_0^A 6 r^5 e^{-ar^2}\, dr\right]$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2A^6}{a^2} e^{-aA^2} + \frac{12}{a^2}\left(-\frac{1}{2a}\right)\int_0^A r^4\, d(e^{-ar^2})$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2A^6}{a^2} e^{-aA^2} - \frac{6}{a^3}\left[e^{-ar^2} r^4\Big|_0^A - \int_0^A 4r^3 e^{-ar^2}\, dr\right]$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2A^6}{a^2} e^{-aA^2} - \frac{6A^4}{a^3} e^{-aA^2} + \frac{24}{a^3}\left(-\frac{1}{2a}\right)\int_0^A r^2\, d(e^{-ar^2})$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2A^6}{a^2} e^{-aA^2} - \frac{6A^4}{a^3} e^{-aA^2}$$

$$- \frac{12}{a^4}\left[e^{-ar^2} r^2\Big|_0^A - \int_0^A 2r\, e^{-ar^2}\, dr\right]$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2A^6}{a^2} e^{-aA^2} - \frac{6A^4}{a^3} e^{-aA^2}$$

$$- \frac{12\ A^2}{a^4} e^{-aA^2} + \frac{12}{a^4}\left(-\frac{1}{2a}\right)\int_0^A 2\, d\, (e^{-ar^2})$$

$$= -\frac{A^8}{2a} e^{-aA^2} - \frac{2A^6}{a^2} e^{-aA^2} - \frac{6A^4}{a^3} e^{-aA^2} - \frac{12\ A^2}{a^4} e^{-aA^2}$$

$$- \frac{12}{a^5}(e^{-aA^2} - 1)$$

$$= \frac{12}{a^5}\left[1 - e^{-aA^2}\left(1 + aA^2 + \frac{a^2A^4}{2} + \frac{a^3A^6}{6} + \frac{a^4A^8}{24}\right)\right] . \qquad \text{(-4)}$$

4. Evaluation of $J_4(A) = \int_0^A r^2 e^{-ar^2} dr$

$$J_4(A) = \int_0^A r^2 e^{-ar^2} dr = -\frac{1}{2a} \int_0^A r \, d(e^{-ar^2})$$

$$= -\frac{1}{2a} \left[e^{-ar^2} r \Big|_0^A - \int_0^A e^{-ar^2} dr \right]$$

$$= -\frac{A}{2a} e^{-aA^2} + \frac{1}{2a} \int_0^A e^{-ar^2} dr .$$

In the integral expression let $\sqrt{a}\, r = \frac{u}{\sqrt{2}}$; thus $dr = \frac{du}{\sqrt{2a}}$. Then

$$J_4(A) = -\frac{A}{2a} e^{-aA^2} + \frac{1}{2a} \int_0^{A\sqrt{2a}} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2a}} .$$

Introducing the error function defined in (12) as

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{w^2}{2}} dw \qquad (B-5)$$

one obtains for $J_4(A)$

$$J_4(A) = -\frac{A}{2a} e^{-aA^2} + \frac{\sqrt{\pi}}{2a^{3/2}} \Phi(A\sqrt{2a}) . \qquad (B-6)$$

5. Evaluation of $J_5(A) = \int_0^A r^6 e^{-ar^2} dr$

$$J_5(A) = \int_0^A r^6 e^{-ar^2} dr = -\frac{1}{2a} \int_0^A r^5 \, d(e^{-ar^2})$$

$$= -\frac{1}{2a} \left[e^{-ar^2} r^5 \Big|_0^A - \int_0^A 5 r^4 e^{-ar^2} dr \right]$$

$$= -\frac{A^5}{2a} e^{-aA^2} + \frac{5}{2a} \int_0^A r^4 e^{-ar^2} dr$$

$$J_5(A) = -\frac{A^5}{2a} e^{-aA^2} + \frac{5}{2a}\left(-\frac{1}{2a}\right) \int_0^A r^3 \, d(e^{-ar^2})$$

$$= -\frac{A^5}{2a} e^{-aA^2} - \frac{5}{4a^2} \left[e^{-ar^2} r^3 \Big|_0^A - \int_0^A 3 r^2 e^{-ar^2} dr \right]$$

$$= -\frac{A^5}{2a} e^{-aA^2} - \frac{5}{4} \frac{A^3}{a^2} e^{-aA^2} + \frac{15}{4a^2} \int_0^A r^2 e^{-ar^2} dr$$

$$= -\frac{A^5}{2a} e^{-aA^2} - \frac{5}{4} \frac{A^3}{a^2} e^{-aA^2} + \frac{15}{4a^2}\left(-\frac{1}{2a}\right) \int_0^A r \, d(e^{-ar^2})$$

$$= -\frac{A^5}{2a} e^{-aA^2} - \frac{5}{4} \frac{A^3}{a^2} e^{-aA^2}$$

$$- \frac{15}{8a^3} \left[e^{-ar^2} r \Big|_0^A - \int_0^A e^{-ar^2} dr \right]$$

$$= -\frac{A^5}{2a} e^{-aA^2} - \frac{5A^3}{4a^2} e^{-aA^2} - \frac{15A}{8a^3} e^{-aA^2} + \frac{15}{8a^3} \int_0^A e^{-ar^2} dr.$$

Making the change in variable $\sqrt{a}\, r = u/\sqrt{2}$ in the last integral yields

$$J_5(A) = -\frac{A^5}{2a} e^{-aA^2} - \frac{5A^3}{4a^2} e^{-aA^2} - \frac{15A}{8a^3} e^{-aA^2} + \frac{15\sqrt{\pi}}{8a^{7/2}} \frac{1}{\sqrt{2\pi}} \int_0^{A\sqrt{2a}} e^{-\frac{u^2}{2}} du.$$

Using expression (B-5) one obtains

$$J_5(A) = \frac{15\sqrt{\pi}}{8a^{7/2}} \Phi(A\sqrt{2a}) - \frac{15}{8} \frac{A}{a^3} e^{-aA^2} \left(1 + \frac{2}{3} aA^2 + \frac{4}{15} a^2A^4\right). \quad \text{(B-7)}$$

6. <u>Evaluation of $J_6(A) = \int_0^A r^{10} e^{-ar^2} dr$</u>

$$J_6(A) = \int_0^A r^{10} e^{-ar^2} dr = -\left(\frac{1}{2a}\right) \int_0^A r^9 \, d(e^{-ar^2})$$

$$= -\frac{1}{2a}\left[e^{-ar^2} r^9 \Big|_0^A - \int_0^A 9\, r^8 e^{-ar^2}\, dr \right]$$

$$J_6(A) = -\frac{A^9}{2a} e^{-aA^2} + \frac{9}{2a}\left(-\frac{1}{2a}\right) \int_0^A r^7\, d(e^{-ar^2})$$

$$= -\frac{A^9}{2a} e^{-aA^2} - \frac{9}{4a^2}\left[e^{-ar^2} r^7 \Big|_0^A - \int_0^A 7\, r^6 e^{-ar^2}\, dr \right]$$

$$= -\frac{A^9}{2a} e^{-aA^2} - \frac{9A^7}{4a^2} e^{-aA^2} + \frac{63}{4a^2} \int_0^A r^6 e^{-ar^2}\, dr$$

$$= -\frac{A^9}{2a} e^{-aA^2} - \frac{9A^7}{4a^2} e^{-aA^2} + \frac{63}{4a^2} J_5(A)$$

$$= -\frac{A^9}{2a} e^{-aA^2} - \frac{9A^7}{4a^2} e^{-aA^2} + \frac{63}{4a^2}\left[\frac{15\sqrt{\pi}}{8a^{7/2}} \Phi(A\sqrt{2a}) \right.$$

$$\left. - \frac{15}{8} \frac{A}{a^3} e^{-aA^2} \left(1 + \frac{2}{3} aA^2 + \frac{4}{15} a^2A^4\right)\right]$$

$$= \frac{945\sqrt{\pi}}{32\, a^{11/2}} \Phi(A\sqrt{2a}) - \frac{945}{32} \frac{A}{a^5} e^{-aA^2} \cdot$$

$$\left(1 + \frac{2}{3} aA^2 + \frac{1}{15} a^2A^4 + \frac{8}{105} a^3A^6 + \frac{16}{945} a^4A^8\right) . \qquad (3\text{-}8)$$

APPENDIX C

EVALUATION OF THE DERIVATIVES OF P_{kb}

The purpose of this appendix is to derive expressions (49) to (52). For convenience, these expressions are repeated below. Let

$$X_n(U) = \int_0^U r^n e^{-ar^2} I_0 (br^2) \, dr \tag{C-1}$$

$$Y_n(U) = \int_0^U r^n e^{-ar^2} I_1 (br^2) \, dr \, . \tag{C-2}$$

Then, expressions (49) to (52) which are to be proved are

$$\frac{\partial P_{kb}}{\partial A} = \frac{P_{kb}}{B-A} - \frac{X_1(A)}{\sigma_x \sigma_y (B-A)} \tag{49}$$

$$\frac{\partial P_{kb}}{\partial B} = \frac{X_1(B)}{\sigma_x \sigma_y (B-A)} - \frac{P_{kb}}{B-A} \tag{50}$$

$$\frac{\partial P_{kb}}{\partial \sigma_x} = - \frac{1}{2\sigma_x^2 \sigma_y (B-A)} \left[X_2(B) - X_2(A) + Y_2(B) - Y_2(A) \right] \tag{51}$$

$$\frac{\partial P_{kb}}{\partial \sigma_y} = \frac{1}{2\sigma_x \sigma_y^2 (B-A)} \left[Y_2(B) - Y_2(A) - X_2(B) + X_2(A) \right] \tag{52}$$

The derivation is divided into five sections. In Section I, some recurrence relations are derived which will be found useful in Sections IV and V. Section II to V provide, respectively, the derivations of (49), (50), (51), and (52).

<u>Section I.</u> <u>Some recurrence relations</u>

Using Leibniz's rule and applying it to (C-1) it is found that

$$\frac{\partial X_n(U)}{\partial a} = -\int_0^U r^{n+2}\, e^{-ar^2}\, I_0(br^2)\, dr \tag{C-3}$$

The integral portions of (C-3) is recognized as being $X_{n+2}(U)$, hence

$$\frac{\partial X_n(U)}{\partial a} = -X_{n+2}(U)\ . \tag{C-4}$$

In a similar fashion one may differentiate (C-1) with respect to b to obtain

$$\frac{\partial X_n(U)}{\partial b} = \int_0^U r^{n+2}\, e^{-ar^2}\, I_1(br^2)\, dr\ . \tag{C-5}$$

This comes about owing to the manner in which $I_0(\phi(x))$ is differentiated, namely,

$$\frac{dI_0(\phi(x))}{dx} = I_1(\phi(x))\,\frac{d\phi(x)}{dx}\ . \tag{C-6}$$

The final result is

$$\frac{\partial X_n(U)}{\partial b} = Y_{n+2}(U)\ . \tag{C-7}$$

The definitions of a and b that are given by (9) and (10) may readily be rearranged to give

$$2(a-b) = \frac{1}{\sigma_x^2} \tag{C-8}$$

and

$$2(a+b) = \frac{1}{\sigma_y^2}\ . \tag{C-9}$$

Appropriate differentiation of (9) and (10) yields

$$\frac{\partial a}{\partial \sigma_x} = -\frac{1}{2\sigma_x^3} \tag{C-10}$$

$$\frac{\partial b}{\partial \sigma_x} = \frac{1}{2\sigma_x^3} \tag{C-11}$$

$$\frac{\partial a}{\partial \sigma_y} = -\frac{1}{2\sigma_y^3} \tag{C-12}$$

and
$$\frac{\partial b}{\partial \sigma_y} = -\frac{1}{2\sigma_y^3} \,. \tag{C-13}$$

Some essential identities are now generated. The rule for integration by parts is given by

$$\int u\, dv = uv - \int v\, du \,. \tag{C-14}$$

In (C1) let

$$r^n dr = du \tag{C-15}$$

and

$$e^{-ar^2} I_0(br^2) = u \,. \tag{C-16}$$

It follows that

$$v = \frac{r^{n+1}}{n+1} \tag{C-17}$$

and

$$du = \left[2br\, e^{-ar^2} I_1(br^2) - 2ar\, e^{-ar^2} I_0(br^2)\right] dr \,. \tag{C-18}$$

Substitution into (C14) gives, upon imposing the limits given in (C-1):

$$X_n(U) = \frac{U^{n+1}\, e^{-aU^2} I_0(bU^2)}{n+1} + \frac{2a}{n+1} X_{n+2}(U) - \frac{2b}{n+1} Y_{n+2}(U) \,. \tag{C-19}$$

A multiplication by (n+1) then gives

$$(n+1)\, X_n(U) = U^{n+1} e^{-aU^2} I_0(bU^2) + 2aX_{n+2}(U) - 2bY_{n+2}(U). \tag{C-20}$$

By taking

$$r^n\ dr = dv \tag{C-21}$$

and

$$e^{-ar^2}\ I_1(br^2) = u \tag{C-22}$$

one may also integrate (C-2) by parts. In this case

$$v = \frac{r^{n+1}}{n+1} \tag{C-23}$$

and

$$du = [2br\ e^{-ar^2}\ I_0(br^2) - e^{-ar^2}\ I_1(br^2)(\tfrac{2}{r} + 2ar)]\ dr. \tag{C-24}$$

The latter relation follows since

$$\frac{dI_1(\phi(x))}{dx} = [I_0\ (\phi(x)) - \frac{I_1(\phi(x))}{\phi(x)}]\ \frac{d\phi(x)}{dx}\ . \tag{C-25}$$

After minor algebraic manipulation the integration by parts gives

$$(n-1)\ Y_n(U) = U^{n+1}\ e^{-aU^2}\ I_1(bU^2) + 2aY_{n+2}(U) - 2bX_{n+2}(U)\ . \tag{C-26}$$

The sum of (C-20) and (C-26) yields

$$\begin{aligned} 2(a-b)\ [Y_{n+2}(U) + X_{n+2}(U)] &= (n+1)\ X_n(U) \\ &+ (n-1)\ Y_n(U) - U^{n+1}\ e^{-aU^2}\ [I_0(bU^2) + I_1(bU^2)] \end{aligned} \tag{C-27}$$

while their difference yields

$$\begin{aligned} 2(a+b)\ [Y_{n+2}(U) - X_{n+2}(U)] &= (n-1)\ Y_n(U) \\ &- (n+1)\ X_n(U) + U^{n+1}\ e^{-aU^2}\ [I_0(bU^2) - I(bU^2)]\ . \end{aligned} \tag{C28}$$

Identities (C-27) and (C-28) will prove to be important in computing (51) and (52).

Section II. Computation of $\frac{\partial P_{kb}}{\partial A}$

By incorporating (C-1) into (11) one obtains

$$P_{kb} = \frac{BX_1(B) - AX_1(A) - X_2(B) + X_2(A)}{(B-A)\,\sigma_x\sigma_y} \tag{C-29}$$

$E[P_{kb}]$ in Section IV-2 may also be calculated using (C-29). Rearranging one obtains

$$(B-A)\,P_{kb} = \frac{BX_1(B) - AX_1(A) - X_2(B) + X_2(A)}{\sigma_x\sigma_y} \tag{C-30}$$

Now differentiate (C-30) with respect to A. Proceeding formally one gets

$$(B-A)\,\frac{\partial P_{kb}}{\partial A} - P_{kb} = \frac{-A\,\frac{\partial X_1(A)}{\partial A} - X_1(A) + \frac{\partial X_2(A)}{\partial A}}{\sigma_x\sigma_y} \tag{C-31}$$

From Leibniz' rule one gets

$$\frac{\partial X_1(A)}{\partial A} = A\,e^{-aA^2}\,I_0(bA^2) \tag{C-32}$$

and

$$\frac{\partial X_2(A)}{\partial A} = A^2\,e^{-aA^2}\,I_0(bA^2)\ . \tag{C-33}$$

Substitution into (C-31) gives

$$(B-A)\,\frac{\partial P_{kb}}{\partial A} - P_{kb} = -\frac{X_1(A)}{\sigma_x\sigma_y} \tag{C-34}$$

or in rearranged form

$$\frac{\partial P_{kb}}{\partial A} = \frac{\sigma_x\sigma_y P_{kb} - X_1(A)}{(B-A)\,\sigma_x\sigma_y} \tag{C-35}$$

which is the same as (49).

Section III. Computation of $\frac{\partial P_{kb}}{\partial B}$

Returning to (C-30) and differentiating with respect to B gives

$$(B-A)\,\frac{\partial P_{kb}}{\partial B} + P_{kb} = \frac{B\,\frac{\partial X_1(B)}{\partial B} + X_1(B) - \frac{\partial X_2(B)}{\partial B}}{\sigma_x \sigma_y}\,. \qquad \text{(C-36)}$$

Since by Leibniz' rule

$$\frac{\partial X_1(B)}{\partial B} = Be^{-aB^2}\, I_0(bB^2) \qquad \text{(C-37)}$$

and

$$\frac{\partial X_2(B)}{\partial B} = B^2 e^{-aB^2}\, I_0(bB^2) \qquad \text{(C-38)}$$

one finds that

$$\frac{\partial P_{kb}}{\partial B} = \frac{X_1(B) - \sigma_x \sigma_y P_{kb}}{(B-A)\,\sigma_x \sigma_y}\,. \qquad \text{(C-39)}$$

This is the same as (50).

Section IV. Computation of $\frac{\partial P_{kb}}{\partial \sigma_x}$

The computation of $\frac{\partial P_{kb}}{\partial \sigma_x}$ is more involved. Start with (C-30) and differentiate to get

$$(B-A)\,\frac{\partial P_{kb}}{\partial \sigma_x} = \frac{1}{\sigma_x \sigma_y}\,\frac{\partial}{\partial \sigma_x}\left[BX_1(B) - AX_1(A) - X_2(B) + X_2(A)\right]$$

$$- \frac{1}{\sigma_x^2 \sigma_y}\left[BX_1(B) - AX_1(A) - X_2(B) + X_2(A)\right]\,. \qquad \text{(C-40)}$$

By the chain rule

$$\frac{\partial(\cdot)}{\partial\sigma_x} = \frac{\partial(\cdot)}{\partial a}\frac{\partial a}{\partial\sigma_x} + \frac{\partial(\cdot)}{\partial b}\frac{\partial b}{\partial\sigma_x} \tag{C-41}$$

and owing to (C-4), (C-7), (C-10), and (C-11) one gets

$$\frac{\partial X_n(U)}{\partial\sigma_x} = \frac{1}{2\sigma_x^3}\left[X_{n+2}(U) + Y_{n+2}(U)\right] . \tag{C-42}$$

A similar calculation gives

$$\frac{\partial X_n(U)}{\partial\sigma_y} = \frac{1}{2\sigma_y^3}\left[X_{n+2}(U) - Y_{n+2}(U)\right] . \tag{C-43}$$

The complicated substitution into (C-40) gives

$$(B-A)\frac{\partial P_{kb}}{\partial\sigma_x} = \frac{1}{2\sigma_x^4\sigma_y}\left[B(X_3(B) + Y_3(B))\right.$$

$$\left. - A(X_3(A) + Y_3(A)) - (X_4(B) + Y_4(B)) + (X_4(A) + Y_4(A))\right]$$

$$- \frac{(B-A)\ P_{kb}}{\sigma_x} . \tag{C-44}$$

Multiplying by $\frac{\sigma_x}{B-A}$ and then invoking (C-8) gives

$$\sigma_x\frac{\partial P_{kb}}{\partial\sigma_x} = -P_{kb} + \left[\frac{1}{2(B-A)\ \sigma_x\sigma_y}\right]\left[2(a-b)\right] \cdot$$

$$\left[B(X_3(B) + Y_3(B)) - A(X_3(A) + Y_3(A)) - (X_4(B) + Y_4(B))\right.$$

$$\left. + (X_4(A) + Y_4(A))\right] . \tag{C-45}$$

With n set to 1 and then 2 in (C-27) and these results inserted into (C-45) one gets after some nice cancellation

$$\sigma_x \frac{\partial P_{kb}}{\partial \sigma_x} = - P_{kb} + \frac{1}{2(B-A)\ \sigma_x \sigma_y} \left[2BX_1(B) - 2AX_1(A) \right.$$

$$\left. - 3X_2(B) - Y_2(B) + 3X_2(A) + Y_2(A) \right] . \qquad \text{(C-46)}$$

Replacing P_{kb} by (C-29) yields after cancellation

$$\sigma_x \frac{\partial P_{kb}}{\partial \sigma_x} = \frac{1}{2(B-A)\ \sigma_x \sigma_y} \left[- X_2(B) - Y_2(B) + X_2(A) + Y_2(A) \right] . \qquad \text{(C-47)}$$

Division by σ_x then gives

$$\frac{\partial P_{kb}}{\partial \sigma_x} = - \frac{\left[X_2(B) - X_2(A) \right] + \left[Y_2(B) - Y_2(A) \right]}{2(B-A)\ \sigma_x^2 \sigma_y} . \qquad \text{(C-48)}$$

This is the same as (51).

Section V. Computation of $\frac{\partial P_{kb}}{\partial \sigma_y}$

The computation of $\frac{\partial P_{kb}}{\partial \sigma_y}$ follows along the same lines, but uses (C-43), (C-9), and (C-28) rather than (C-42), (C-8), and (C-27). The result is

$$\frac{\partial P_{kb}}{\partial \sigma_y} = - \frac{\left[X_2(B) - X_2(A) \right] - \left[Y_2(B) - Y_2(A) \right]}{2(B-A)\ \sigma_x \sigma_y^2} \qquad \text{(C-49)}$$

which is the same as (52).

APPENDIX D

DEVELOPMENT OF EXPRESSIONS FOR $X_m(U)$ AND $Y_m(U)$.

In this appendix, relations (54) and (55) are derived. First expression (54) is developed. Define

$$X_n(U) = \int_0^U r^n e^{-ar^2} I_0(br^2)\, dr \qquad \text{(D-1)}$$

and make the change in variable

$$\rho = ar^2; \qquad \text{(D-2)}$$

this gives

$$X_n(U) = \frac{1}{2a^{\frac{n+1}{2}}} \int_0^{aU^2} \rho^{\frac{n-1}{2}} e^{-\rho} I_0\left(\frac{b}{a}\rho\right) d\rho \; . \qquad \text{(D-3)}$$

The Maclaurin series expansion for $I_0\left(\frac{b}{a}\rho\right)$ is

$$I_0\left(\frac{b}{a}\rho\right) = \sum_{k=0}^{\infty} \left(\frac{b}{a}\right)^{2k} \frac{\rho^{2k}}{(k!)^2 2^{2k}} \; . \qquad \text{(D-4)}$$

Substitution into (D-3) followed by interchange of the order of integration and summation gives

$$X_n(U) = \frac{1}{2a^{\frac{n+1}{2}}} \sum_{k=0}^{\infty} \frac{\left(\frac{b}{a}\right)^{2k}}{(k!)^2 2^{2k}} \int_0^{aU^2} e^{-\rho} \rho^{\frac{4k+n-1}{2}} d\rho \; . \qquad \text{(D-5)}$$

The integral portion of (D-5) is recognized as an incomplete gamma function and may be evaluated using the tables of the Chi-Squared Probability function expressed in (53). To see this, consider the integral

$$P(\alpha,x) = \frac{1}{\Gamma(\alpha)} \int_0^x e^{-\rho} \rho^{\alpha-1} d\rho \ . \tag{D-6}$$

Let

$$x = \frac{\chi^2}{2} \tag{D-7}$$

and $$\alpha = \frac{\nu}{2} \ . \tag{D-8}$$

This allows (D-6) to be written as

$$P((\frac{\nu}{2}, \frac{\chi^2}{2}) = \frac{1}{\Gamma(\frac{\nu}{2})} \int_0^{\frac{\chi^2}{2}} e^{-\rho} \rho^{\frac{\nu}{2} - 1} d\rho \ . \tag{D-9}$$

Upon replacing ρ by $\frac{t}{2}$ yields

$$P(\frac{\nu}{2}, \frac{\chi^2}{2}) = \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^{\chi^2} e^{-\frac{t}{2}} (t)^{\frac{\nu}{2} - 1} dt$$

$$= P(\chi^2|\nu) \tag{D-10}$$

as defined in (53). Hence (D-6) may be written as

$$P(\alpha,x) = P(2x|2\alpha) \ .$$

Comparison of (D-6) with (D-5) indicates that

$$2\alpha = 4k+n+1 \tag{D-11}$$

and $$2x = 2a\mu^2 \ . \tag{D-12}$$

Hence (D-5) may be written as

$$X_n(U) = \frac{1}{2a^{\frac{n+1}{2}}} \sum_{k=0}^{\infty} \frac{\left(\frac{b}{a}\right)^{2k} \Gamma\left(\frac{4k+n+1}{2}\right) P(2aU^2|4k+n+1)}{(k!)^2\, 2^{2k}} . \qquad \text{(D-13)}$$

But owing to the properties of the gamma function

$$\Gamma\left(\frac{4k+n+1}{2}\right) = \frac{(n+1)(n+3)\cdots(n+4k-1)}{2^{2k}} \Gamma\left(\frac{n+1}{2}\right) \qquad \text{(D-14)}$$

for k=1,2,...; and, hence,

$$X_n(U) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}} \left[P(2aU^2|n+1)\right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{\left(\frac{b}{4a}\right)^{2k} (n+1)(n+3)\cdots(4k+n-1)}{(k!)^2} P(2aU^2|4k+n+1)\right] . \qquad \text{(D-15)}$$

This is the same as (54).

The development of (55) follows along the same lines except that in (D-3) $X_n(U)$ is replaced by $Y_n(U)$, as defined in (48), and $I_0(\frac{b}{a}\rho)$ is replaced by $I_1(\frac{b}{a}\rho)$ where

$$I_1\left(\frac{b}{a}\rho\right) = \sum_{k=0}^{\infty} \left(\frac{b}{a}\right)^{2k+1} \frac{\rho^{2k+1}}{(k)!\,(k+1)!\,2^{2k+1}} . \qquad \text{(D-16)}$$

The final result is the same as (55) or

$$Y_n(U) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{\frac{n+1}{2}}} \sum_{k=0}^{\infty} \frac{\left(\frac{b}{4a}\right)^{2k+1} (n+1)(n+3)\cdots(4k+n+1)}{k!\,(k+1)!} P(2aU^2|4k+n+3) \qquad \text{(D-17)}$$

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